PS8

8.1

2 a) Find a recurrence relation for the number of permu–tations of a set with *n* elements.

b) Use this recurrence relation to find the number of per–mutations of a set with *n* elements using iteration.

10 a) Find a recurrence relation for the number of bit strings of length *n* that contain the string 01.

b) What are the initial conditions?

c) How many bit strings of length seven contain the string 01?

15 a) Find a recurrence relation for the number of ternary strings of length *n* that do not contain two consecutive 0s or two consecutive 1s.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain two consecutive 0s or two consecutive 1s?

32 In the Tower of Hanoi puzzle, suppose our goal is to transfer all n disks from peg 1 to peg 3, but we cannot move a disk directly between pegs 1 and 3 . Each move of a disk must be a move involving peg 2. As usual, we cannot place a disk on top of a smaller disk.

a) Find a recurrence relation for the number of moves required to solve the puzzle for n disks with this added restriction.

b) Solve this recurrence relation to find a formula for the number of moves required to solve the puzzle for n disks.

c) How many different arrangements are there of the n disks on three pegs so that no disk is on top of a smaller disk?

d) Show that every allowable arrangement of the n disks occurs in the solution of this variation of the puzzle.

8.2

1 Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

b) $a_n = 2na_{n-1} + a_{n-2}$

c) $a_n = a_{n-1} + a_{n-4}$

d) $a_n = a_{n-1} + 2$

e) $a_n = a_{n-1}^2 + a_{n-2}$

f) $a_n = a_{n-2}$

g) $a_n = a_{n-1} + n$

3 Solve these recurrence relations together with the initial conditions given.

a) $a_n = 2a_{n-1}$ for $n \geq 1, a_0 = 3$

b) $a_n = a_{n-1}$ for $n \geq 1, a_0 = 2$

c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2, a_0 = 1, a_1 = 0$

d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 6, a_1 = 8$

e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 0, a_1 = 1$

f) $a_n = 4a_{n-2}$ for $n \geq 2, a_0 = 0, a_1 = 4$

g) $a_n = a_{n-2}/4$ for $n \geq 2, a_0 = 1, a_1 = 0$

18 Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5, a_1 = 4$, and $a_2 = 88$.

40 Solve the simultaneous recurrence relations

$$a_n = 3a_{n-1} + 2b_{n-1}$$
$$b_n = a_{n-1} + 2b_{n-1}$$

with $a_0 = 1$ and $b_0 = 2$.

8.3 Suppose that $f(n) = 2f(n/2) + 3$ when n is an even positive integer, and $f(1) = 5$. Find

a) $f(2)$

b) $f(8)$

c) $f(64)$

d) $f(1024)$

14 Suppose that there are $n = 2^k$ teams in an elimination tournament, where there are $n/2$ games in the first round, with the $n/2 = 2^{k-1}$ winners playing in the second round, and so on. Develop a recurrence relation for the number of rounds in the tournament.

18 Suppose that each person in a group of n people votes for exactly two people from a slate of candidates to fill two positions on a committee. The top two finishers both win positions as long as each receives more than n / 2 votes.

a) Devise a divide-and-conquer algorithm that determines whether the two candidates who received the most votes each received at least n / 2 votes and, if so, determine who these two candidates are.

b) Use the master theorem to give a big- O estimate for the number of comparisons needed by the algorithm you devised in part (a).

28 Suppose someone picks a number x from a set of n numbers. A second person tries to guess the number by successively selecting subsets of the n numbers and asking the first person whether x is in each set. The first person answers either "yes" or "no." When the first person answers each query truthfully, we can find x using $logn$ queries by successively splitting the sets used in each query in half. Ulam's problem, proposed by Stanislaw Ulam in 1976, asks for the number of queries required to find x, supposing that the first person is allowed to lie exactly once.

a) Show that by asking each question twice, given a number x and a set with n elements, and asking one more question when we find the lie, Ulam's problem can be solved using 2 $logn+1$ queries.

b) Show that by dividing the initial set of n elements into four parts, each with n / 4 elements, 1 / 4 of the elements can be eliminated using two queries. [Hint: Use two queries, where each of the queries asks whether the element is in the union of two of the subsets with n / 4 elements and where one of the subsets of n / 4 elements is used in both queries.]

c) Show from part (b) that if $f(n)$ equals the number of queries used to solve Ulam's problem using the method from part (b) and n is divisible by 4 , then $f(n)=f(3n/4)+2$.

d) Solve the recurrence relation in part (c) for $f(n)$.

e) Is the naive way to solve Ulam's problem by asking each question twice or the divide-and-conquer method based on part (b) more efficient? The most efficient way to solve Ulam's problem has been determined by A. Pelc [Pe87].

In Exercises 29-33, assume that f is an increasing function satisfying the recurrence relation $f(n)=af(n/b)+cn^d$, where $a \geq 1, b$ is an integer greater than 1 , and c and d are positive real numbers. These exercises supply a proof of Theorem 2.

8.5

1 How many elements are in $A_1 \cup A_2$ if there are 12 elements in $A_1, 18$ elements in A_2 , and

a) $A_1 \cap A_2=\emptyset$?

b) $|A_1 \cap A_2|=1$?

c) $|A_1 \cap A_2| = 6$?

d) $A_1 \subseteq A_2$?

4 A marketing report concerning personal computers states that 650,000 owners will buy a printer for their machines next year and 1,250,000 will buy at least one software package. If the report states that 1,450,000 owners will buy either a printer or at least one software package, how many will buy both a printer and at least one software package?

16 How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish*,*rat* or *bird*?

26 Find the probability that when a fair coin is flipped five times tails comes up exactly three times, the first and last flips come up tails, or the second and fourth flips come up heads

TEST1

1 Find a recurrence relation and initial condition for the number of fruit flies in a jar if there are 12 flies initially and every week there are six times as many flies in the jar as there were the previous week.

2 Find the solution of the recurrence relation $a_n = 3a_{n-1}$, with $a_0 = 2$.

3 Find the solution of the linear homogeneous recurrence relation $a_n = 7a_{n-1} - 6a_{n-2}$ with $a_0 = -1$ and $a_1 = 4$.

4 Suppose that f(n) satisfies the divide–and–conquer relation f(n)=2 f(n / 3)+5 and f(1)=7.What is f(81) ?

5 Suppose that $|A| = |B| = |C| = 100, |A \cap B| = 60, |A \cap C| = 50, |B \cap C| = 40$, and $|A \cup B \cup C| = 175$. How many elements are in $A \cap B \cap C$?

TEST2

1 (a) Find a recurrence relation for the number of ways to climb n stairs if stairs can be climbed two or three at a time.

(b) What are the initial conditions?

(c) How many ways are there to climb eight stairs?

2 What is the solution to the recurrence relation $a_n = 8a_{n-1} + 9a_{n-2}$ if $a_0 = 3$ and $a_1 = 7$?

3 Suppose that $f(n) satisfies the divide–and–conquer recurrence relation $f(n) = 3f(n/4) + n^2/8$ with f(1)=2. What is f(64) ?